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AdS/CFT, multitrace deformations and new instabilities of nonlocal string theories

José L. F. Barbón*
Theory Division, CERN, CH-1211 Geneva 23, Switzerland
E-mail: barbon@cern.ch

Carlos Hoyos
Instituto de Física Teórica UAM/CSIC, C-XVI
and Departamento de Física Teórica, C-XI
Universidad Autónoma de Madrid, E-28049–Madrid, Spain
E-mail: c.hoyos@uam.es

Abstract: We study "multitrace" deformations of large-N master fields in models with a mass gap. In particular, we determine the conditions for the multitrace couplings to drive tachyonic instabilities. These tachyons represent new local instabilities of the associated nonlocal string theories. In the particular case of Dp-branes at finite temperature, we consider topology-changing phase transitions and the effect of multitrace perturbations on the corresponding phase diagrams.

Keywords: 1/N Expansion, Superstring Vacua, Brane Dynamics in Gauge Theories.

*On leave from Departamento de Física de Partículas da Universidade de Santiago de Compostela, Spain.

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Contents

1. Introduction

The AdS/CFT correspondence [1, 2] provides a nonperturbative definition of quantum gravity in asymptotically anti-de Sitter (AdS) spaces, in terms of a quantum field theory with an ultraviolet fixed point (CFT), formally living on a conformal boundary of the AdS space. The prototype examples define string theory on certain AdS_{d+1} vacua as dual to supersymmetric $\mathrm{SU}(N)$ Yang–Mills (SYM) theories in the large-N limit of 't Hooft [3], defined on $\mathbb{R} \times \mathbf{S}^d$. In this correspondence, string perturbation theory is obtained as the $1/N$ expansion of the gauge theory, and different semiclassical backgrounds with a common AdS asymptotics correspond to different "master fields" of the gauge theory [4].

Single-string states in perturbation theory arise as single-trace states in the Yang–Mills theory, such as $W_\gamma \left|\mathrm{vac}\right\rangle$, where

$$W_\gamma = \mathrm{tr}\,\mathrm{P}\,\exp\left(i\oint_\gamma A\right) \tag{1.1}$$

for some contour γ. For CFT's one has an exact operator-state mapping, so that we may equivalently talk about the space of local operators of the CFT on $\mathbb{R}^d$. For finite values of N, i.e. at the nonperturbative level, this correspondence between single strings and single traces must break down, since the Yang–Mills theory with gauge group of rank N has only $O(N)$ independent single-trace *local* operators. Many single-trace local operators are linearly related to multitrace local operators, which means that the perturbative Fock space of string perturbation theory loses its exact meaning for excitations with $O(N)$ strings. This is related to the so-called "stringy exclusion principle" [5].

One way of studying nonperturbative effects associated with these phenomena is to consider deformations of the AdS/CFT correspondence by "condensates" of multistring states in the AdS space, i.e. deformations of the CFT by multitrace operators. Such backgrounds were studied in [6] and argued to provide new classes of string theories with nonlocal interactions, both on the worldsheet and the target space (NLST). A nontrivial generalization of the AdS/CFT correspondence is needed to describe these deformations in the leading large-N approximation [7, 8, 9].

It was shown in [8, 10] that these AdS/CFT prescriptions are equivalent to the Hartree approximation from the point of view of the gauge theory, in the spirit of [11]. The basic gauge invariants are single-trace normalized operators of the form $\mathcal{O}_n = N^{-1}\,\mathrm{tr}\,F^n$. Interactions that are non-linear on the basic invariants, such as

$$\mathcal{L}_{\rm int} \sim c_m\,(\mathcal{O}_n)^m\,, \tag{1.2}$$

may be substituted by a linear interaction

$$\mathcal{L}_{\rm eff} \sim C_n\,\mathcal{O}_n\,, \tag{1.3}$$

with C_n a "collective potential" that is determined selfconsistently. The basic identity determining C_n is the saddle-point equation at $N=\infty$. The expectation value of the field equation for $\mathcal{O}_n$ gets a contribution from (1.2) of the form

$$m\,c_m\langle(\mathcal{O}_n)^{m-1}\rangle \longrightarrow m\,c_m\,(\langle\mathcal{O}_n\rangle)^{m-1}\,, \tag{1.4}$$

where we have used the large-N factorization of gauge-invariant operators in the right hand side. Since $\langle\,\mathcal{O}_n\rangle$ are pure numbers, this is the contribution of a linear action (1.3) to the field equation, provided we adjust the "effective potential" according to

$$C_n = m\,c_m\,(\langle\mathcal{O}_n\rangle)^{m-1}\,. \tag{1.5}$$

In the linear model (1.3) we may calculate the expectation values $\langle\mathcal{O}_n\rangle$ by taking derivatives with respect to C_n, so that the expectation values are functions of C_n, and (1.5) becomes an equation that determines C_n selfconsistently.

More generally, for an arbitrary action functional with 't Hooft's scaling

$$S = N^2\int d^dx\,L(\,\mathcal{O}_1,\mathcal{O}_2,\ldots)\,, \tag{1.6}$$

we can introduce an effective single-trace model

$$\overline{S} = N^2\sum_n\int d^dx\,\overline{\zeta}_n\,\mathcal{O}_n\,, \tag{1.7}$$

where the effective couplings $\overline{\zeta}$ are determined by the solution of the "master equations":

$$\overline{\zeta}_n = \frac{\partial L}{\partial \mathcal{O}_n} \left(\langle \mathcal{O} \rangle_{\overline{\zeta}} \right) . \tag{1.8}$$

When interpreted in the context of the AdS/CFT correspondence, (1.8) is equivalent to the boundary conditions proposed in [8].

It was pointed out in [10] that (1.8) can have various solutions for each of the effective couplings ζ_n. Hence, the "master field" of the theory with multitrace interactions can develop a branched structure. On the dominating branch the partition function is maximized or, equivalently, the large-N vacuum energy is minimized. When level-crossing phenomena cause the dominating branch to change, we will have in general large-N phase transitions. These phase transitions correspond to nonperturbative tunneling processes between different backgrounds of the NLST. Similar phenomena were extensively studied in the context of $c \leq 1$ matrix models [12].

In this paper we extend the results of [10] on particular solutions of (1.8) for specific examples. In addition, we study more carefully the structure of large-N phase transitions induced by multitrace perturbations, and we compare their effects with other types of large-N phase transitions that can be calculated using the AdS/CFT correspondence. We study the conditions for the NLST backgrounds to be locally stable, and we find a new kind of tachyonic instabilities triggered by the multitrace interactions.

The article is organized as follows. In section 2 we discuss the general structure of the $1/N$ expansion around a background with multitrace interactions. In section 3 we study the local stability of the saddle points, including the new tachyonic instabilities of multitrace origin and a detailed discussion of one concrete example. In section 4 we study multitrace effects on one particular instance of *global* instabilities, namely topology-changing transitions on toroidal backgrounds, generalizations of the Hawking–Page transition in the pure AdS case [13, 14]. We end with the conclusions and an appendix dealing with subtleties arising in the case of the topological-charge operator in four-dimensional Yang–Mills theories.

2. Systematics of the mean field approximation

Let us consider a gauge-theory model specified by a single-trace action S_0 and a multitrace perturbation by a general function of a single-trace operator of the form

$$\mathcal{O} \sim \frac{1}{N} \operatorname{tr} F^n + \cdots \tag{2.1}$$

where the dots stand for other terms in gauge or matter fields. The complete action is

$$S = S_0 + N^2 \int d^d x \, \zeta \, \mathcal{O} + N^2 \int d^d x \, f(\mathcal{O}) \, , \tag{2.2}$$

where we have separated explicitly the linear part of $f(\mathcal{O})$. We shall assume that operator condensates in this theory, $\langle \mathcal{O} \rangle$, are determined in terms of the microscopic couplings, with at most a discrete degeneracy. In particular, this means that we will only consider theories whose single-trace limit $f \to 0$ has isolated vacua with a mass gap, separated from any possible moduli spaces of vacua.

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The coupling ζ will be promoted to a source for the operator $\mathcal{O}$, although the large-N master field will be assumed translationally invariant on $\mathbb{R}^d$, and ζ will be evaluated as a constant in all expectation values.

In the Hartree approximation, the expectation value $\langle \mathcal{O} \rangle$ may be calculated at $N = \infty$ from the single-trace effective theory with action

$$\overline{S} = S_0 + N^2 \int d^d x \, \overline{\zeta} \, \mathcal{O} \,, \tag{2.3}$$

with $\overline{\zeta}$ determined from the master equation:

$$\overline{\zeta} = \zeta + f' \left(\langle \mathcal{O} \rangle_{\overline{\zeta}} \right) . \tag{2.4}$$

The consideration of large-N phase transitions requires comparing the values of the full partition function

$$\mathcal{Z} [\, \zeta, f \,] = \int DA \, e^{-S} \tag{2.5}$$

at different large-N master fields. The path integral measure over the gauge field A may contain various other fields as well, although we use the notation DA for simplicity. The effective single-trace partition function

$$\overline{\mathcal{Z}} \, [\, \overline{\zeta} \,] = \int DA \, e^{-\overline{S}} \tag{2.6}$$

can be used to compute the single-trace expectation values, but it is in general different from (2.5) at the saddle points.[1]

In order to derive a useful expression for $\mathcal{Z}[\zeta, f]$ it is convenient to introduce appropriate auxiliary fields. First, we insert a delta-functional constraint by the identity

$$\mathbf{1} = \int \prod_x d\sigma_x \, \delta \, [\sigma - \mathcal{O}] \,, \tag{2.7}$$

which defines $\sigma(x)$ as a classical interpolating field for the local operator $\mathcal{O}(x)$. We can further exponentiate the delta-functional by means of a second auxiliary field

$$\delta \, [\sigma - \mathcal{O}] = \int \prod_x \frac{d\chi_x}{2\pi} \, e^{i \int \chi (\sigma - \mathcal{O})} \,. \tag{2.8}$$

It will be convenient to distribute democratically the factor of $(2\pi)^{-1}$ between the σ and χ measures. Defining

$$D\sigma = \prod_x \frac{d\sigma_x}{\sqrt{2\pi}}$$

and analogously for $D\chi$, we can write

$$\mathbf{1} = \int D\sigma \, D\chi \, e^{i \int \chi (\sigma - \mathcal{O})} \,, \tag{2.9}$$

[1] This wrong assumption was made in [10] in a brief analysis of some examples. In the present work we correct these errors, although they do not change the qualitative picture.

which defines a formal path integral representation of the full partition function,

$$\mathcal{Z}[\zeta, f] = \int D\sigma\, D\chi\, \mathcal{Z}_0 \left[\zeta + i\frac{\chi}{N^2}\right] \exp\left[-N^2 \int f(\sigma) + i \int \chi\, \sigma\right]. \tag{2.10}$$

In this expression $\mathcal{Z}_0$ denotes the single-trace partition function that results by setting $f = 0$. The connected functionals of the full and single-trace theory are given by

$$W[\zeta, f] = -\frac{1}{N^2} \log \mathcal{Z}[\zeta, f], \qquad W_0[\zeta] = -\frac{1}{N^2} \log \mathcal{Z}_0[\zeta]. \tag{2.11}$$

In these definitions, ζ is treated as a spacetime-dependent source. At the saddle points we will assume translational invariace on $\mathbb{R}^d$ and it will be useful to define the volume densities

$$W[\zeta, f] = \int d^d x\, w(\zeta, f), \qquad W_0[\zeta] = \int d^d x\, w_0(\zeta), \tag{2.12}$$

as a function of the microscopic couplings.

Both W_0 and all its functional derivatives have a $1/N^2$ expansion with leading term of $O(1)$. They generate the set of connected correlators of $\mathcal{O}(x)$ in the single-trace theory. For example, the one-point function is of $O(1)$,

$$\langle \mathcal{O} \rangle_\zeta = \frac{\delta W_0}{\delta \zeta} = w_0'(\zeta), \tag{2.13}$$

while the connected two-point function is of $O(1/N^2)$,

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle_{c,\zeta} = -\frac{1}{N^2} \frac{\delta W_0}{\delta\zeta(x_1)\, \delta\zeta(x_2)}. \tag{2.14}$$

In order to derive a $1/N$ expansion for $\mathcal{Z}[\zeta, f]$ we evaluate the path integral over the auxiliary fields in the saddle-point approximation. We write $\sigma = \sigma_c + \sigma'/N$ and $\chi = \chi_c + N\chi'$, and we determine σ_c and χ_c requiring the cancellation of the $O(N)$ terms. This leads to the equations

$$\sigma_c = \left.\frac{\delta W_0}{\delta \zeta}\right|_{\zeta + i\chi_c/N^2}, \qquad i\chi_c = N^2 f'(\sigma_c). \tag{2.15}$$

Defining an effective coupling

$$\overline{\zeta} = \zeta + \frac{i\chi_c}{N^2} \tag{2.16}$$

we see that $\sigma_c = \langle \mathcal{O} \rangle_{\overline{\zeta}}$, and the saddle-point equations (2.15) are equivalent to the master equation (2.4). These equations imply that the saddle point of the χ integral lies in the imaginary axis for real values of the one-point function. In this case, one must deform the contour of integration of the zero mode of χ accordingly.

The leading term in the $1/N$ expansion of the partition function is then given by

$$w(\zeta, f) = w_0(\overline{\zeta}) + f\left(\langle \mathcal{O} \rangle_{\overline{\zeta}}\right) - \langle \mathcal{O} \rangle_{\overline{\zeta}}\, f'\left(\langle \mathcal{O} \rangle_{\overline{\zeta}}\right) + O\left(\frac{1}{N^2}\right). \tag{2.17}$$

This shows that the partition function at large-N is not just the partition function of the effective single-trace model; there are two extra terms that were omitted in [10].

The local stability of a given master field is controlled by the functional quadratic in the perturbation fields σ', χ', of $O(1)$ in the large-N expansion around the saddle point. This term contributes

$$-\log \mathcal{Z}[\zeta, f] = O(N^2) + \frac{1}{2}\log \text{Det}\,[\mathcal{K}] + O\left(\frac{1}{N^2}\right), \tag{2.18}$$

where $\mathcal{K}$ is a nonlocal operator acting on the field-space (χ', σ') as

$$\mathcal{K} = \begin{pmatrix} \overline{G}_2 & -i \\ -i & f''_c \end{pmatrix}, \tag{2.19}$$

with $f''_c = f''(\sigma_c)$ and

$$\overline{G}_2(x_1, x_2) = -\frac{\delta^2 W_0}{\delta\,\zeta(x_1)\,\delta\,\zeta(x_2)}\bigg|_{\overline{\zeta}} = N^2\,\langle\,\mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{c,\overline{\zeta}}\,, \tag{2.20}$$

the connected two-point function in the effective single-trace model.

The higher-order functional derivatives of W_0, together with the higher derivatives of the multitrace potential $f(\sigma)$, define effective vertices for the $1/N$ expansion of (2.10). The connected single-trace correlators define nonlocal vertices of the χ' field,

$$V_W = \sum_{n\geq 3}\frac{1}{N^{n-2}}\frac{i^n}{n!}\int d^dx_1\cdots d^dx_n\,\frac{\delta^n W_0}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)}\bigg|_{\overline{\zeta}}\chi'(x_1)\cdots\chi'(x_n)\,, \tag{2.21}$$

where each functional derivative of W_0 has a separate expansion in powers of $1/N^2$ with leading term of $O(1)$.

Both the saddle-point equations (2.15) and the propagator (2.19) depend implicitly on the full single-trace connected functional W_0, which itself has a $1/N^2$ expansion with leading term of $O(1)$. In principle, we have the choice of keeping this implicit $1/N$ expansion in the value of the saddle point σ_c, χ_c and the effective propagator $\mathcal{K}$. However, we often ignore the explicit form of W_0 beyond the planar approximation and, in practice, we solve for σ_c, χ_c just at the leading (planar) order. In this case we may include the nonplanar dependence of (2.15) and (2.19) via a series of explicit tadpole and mass insertions. These new vertices have the form (2.21) with $n = 1, 2$ and W_0 replaced by its "non-planar" part with leading scaling of $O(1/N^2)$. This means that the tadpoles form a series with leading term of $O(1/N)$, whereas the mass insertions start at $O(1/N^2)$.

Finally, the vertices for the σ' field are local,

$$V_\sigma = \sum_{m\geq 3}\frac{1}{N^{m-2}}\frac{1}{m!}\int d^dx\,f^{(m)}(\sigma_c)\,(\sigma'_x)^m\,. \tag{2.22}$$

Using $\mathcal{K}$ as a propagator and (2.21), (2.22) as vertices, we can calculate the $1/N$ corrections to the master field in a systematic diagram technique. For quadratic perturbations, $f''' = 0$, the σ field is free and can be explicitly integrated out, leaving only the diagram technique of the χ field, as in the treatment of [15].

Regarding the NLST interpretation, if the single-trace model is associated to some string background X_0 via AdS/CFT or a deformation of it thereof, the perturbation expansion in powers of the multitrace vertices $f^{(m)}(\sigma_c)$ defines the nonlocal worlsheet interactions, according to [6]. The content of the Hartree approximation is simply that one-point functions and partition functions may be calculated at large-N by working in a modified single-trace background $\overline{X}$, characterized by effective single-trace couplings $\overline{\zeta}$. However, the physics of the NLST goes much beyond the one-point functions and the large-N vacuum energy. In particular, the stability of the NLST cannot be inferred directly from the stability of $\overline{X}$ in the single-trace theory, but instead requires a specific analysis.

3. Stability of the master field

The global stability properties of the perturbed model depend to a large extent on the global properties of the function $f(\sigma)$ for large values of σ. If this function is unbounded from below we can expect a globally unstable model. In this section we shall concetrate on the local stability properties of a given saddle point, characterized by a solution σ_c, χ_c of the master equations (2.15).

In momentum space the operator $\mathcal{K}$ has the block form (2.19) for each value of the momentum. The naive stability conditions demand positivity of the eigenvalues

$$\lambda_k^{\pm} = \frac{1}{2}\left(f_c'' + \overline{G}_2(k)\right) \pm \frac{1}{2}\sqrt{\left((f_c'' - \overline{G}_2(k)\right)^2 - 4}. \tag{3.1}$$

This amounts to the reality condition

$$|f_c'' - \overline{G}_2(k)| > 2\,, \tag{3.2}$$

together with the positivity conditions

$$f_c'' + \overline{G}_2(k) > 0 \qquad \text{and} \qquad 1 + f_c''\,\overline{G}_2(k) > 0\,. \tag{3.3}$$

In ordinary field theories we expect $\overline{G}_2(k) > 0$ for hermitean $\mathcal{O}(x)$, so that the violation of the stability conditions can only occur for sufficiently negative values of f_c''. When these conditions are not met, we can still define the integrals by analytic continuation in f_c'' or, equivalently, by an appropriate contour rotation of the σ', χ' integrals around the saddle point. However, in this process physical quantities will pick up complex phases that change their physical interpretation. One simple example of this phenomenon is the imaginary part of the vacuum energy, which should be interpreted as the total decay width of the unstable saddle point.

Since χ is a formal auxiliary field, it is not obvious that all the conditions $\lambda_k^{\pm} > 0$ have the same physical status. For example, the contribution to the vacuum energy coming from each momentum mode is given by

$$\frac{1}{2}\log\left(\lambda_k^+\lambda_k^-\right) = \frac{1}{2}\log\left[1 + f_c''\,\overline{G}_2(k)\right], \tag{3.4}$$

and simply demanding that this contribution be real imposes the less restrictive condition

$$1 + f_c''\,\overline{G}_2(k) > 0\,. \tag{3.5}$$

In fact, to the extent that we are interested in the exact correlation functions of the operator $\mathcal{O}(x)$, we must focus on the propagation properties of the $\sigma(x)$ field after $\chi(x)$ has been integrated out, since we have

$$\langle\, \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \langle\, \sigma(x_1)\cdots\sigma(x_n)\,\rangle \tag{3.6}$$

as an exact statement in the complete theory. Therefore, we first integrate over χ' and obtain an effective action for σ' of the form

$$\exp\left(-\Gamma_{\rm eff}[\sigma']\right) = {\rm Det}^{-1/2}\left[\overline{G}_2\right]\,\exp\left[-\frac{1}{2}\int \sigma'\,(\,\overline{G}_2^{\,-1} + f_c''\,)\,\sigma' + O\left(\frac{1}{N}\right)\right], \tag{3.7}$$

where we have omitted the interaction terms that are suppressed by powers of $1/N$. Integrating now over σ' in the gaussian approximation we obtain the previous result for the total determinant (3.4),

$${\rm Det}^{-1/2}\left[\overline{G}_2\right]\,\cdot\,{\rm Det}^{-1/2}\left[\overline{G}_2^{\,-1} + f_c''\right] = {\rm Det}^{-1/2}\left[\,1 + f_c''\,\overline{G}_2\right], \tag{3.8}$$

where we assume a convenient regularization procedure to make sense of these manipulations. More generally, the $1/N$ expansion of the correlators (3.6) can be calculated using Feynman rules with effective kinetic term

$$\mathcal{K}_\sigma = \overline{G}_2^{\,-1} + f_c''\,. \tag{3.9}$$

Hence, the physical stability condition demands positivity of this (euclidean) kinetic term for all momenta. Assuming $\overline{G}_2(p) > 0$, this condition is equivalent to (3.5) above.

In order to give a more physical characterization of the stability conditions let us consider the spectral representation of the single-trace two-point function

$$\overline{G}_2(p) = \int dz \frac{\rho(z)}{z+p^2}\,, \tag{3.10}$$

with $\rho(z) > 0$ the spectral density (for simplicity, we assume that $\mathcal{O}(x)$ is scalar and hermitean). Expanding $\overline{G}_2^{\,-1}$ near $p^2 = 0$ we have

$$\overline{G}_2(p) \approx \frac{Z_0(\mathcal{O})}{p^2 + M_0^2}\,, \tag{3.11}$$

where the wave-function rescaling, $Z_0(\mathcal{O})$, and the mass gap, M_0^2, of the single-trace theory are assumed to be positive,

$$Z_0(\mathcal{O}) = \frac{\left(\int dz\,\rho(z)/z\right)^2}{\int dz\,\rho(z)/z^2}\,, \qquad M_0^2 = \frac{\int dz\,\rho(z)/z}{\int dz\,\rho(z)/z^2}\,. \tag{3.12}$$

Hence, the physical stability condition boils down to

$$1 + f_c''\,\overline{G}_2(0) > 0\,. \tag{3.13}$$

The single-trace mass gap gets renormalized and this defines an effective mass gap,

$$M_0^2 \longrightarrow M_{\rm eff}^2 = M_0^2 + Z_0(\mathcal{O})\,f_c'' \geq 0\,, \tag{3.14}$$

that must be positive for stability.

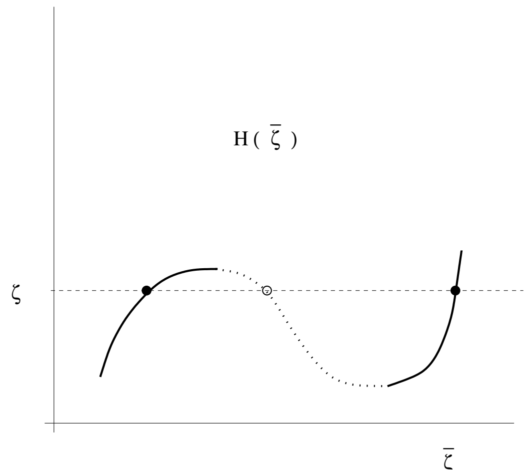

Figure 1: Sketch of a function $H(\overline{\zeta})$ leading to various solutions of the master equation. Stable branches are indicated in solid lines, separated by locally unstable branches in dotted lines. At the extrema of $H(\overline{\zeta})$ we have the onset of the tachyonic instabilities.

3.1 Stability and the master equation

We can relate (3.13) to the master equation by evaluating $\overline{G}_2(0)$. The integrated two-point function at zero momentum can be formally written as

$$\int d^d x\, \overline{G}_2(0) = N^2 \int d^d x \int d^d y\, \langle \mathcal{O}(x)\, \mathcal{O}(y)\rangle_{c,\overline{\zeta}} = -\int d^d x\, w_0''(\overline{\zeta})\,. \qquad (3.15)$$

Hence, the stability condition (3.13) becomes

$$1 - f_c''\, w_0''(\overline{\zeta}) > 0\,. \qquad (3.16)$$

On the other hand, the master equation can be written as $\zeta = H(\overline{\zeta})$, where

$$H(\overline{\zeta}) = \overline{\zeta} - f'\left(w_0'(\overline{\zeta})\right). \qquad (3.17)$$

This function satisfies

$$H'(\overline{\zeta}) = 1 - f_c''\; w_0''(\overline{\zeta})\,, \qquad (3.18)$$

which is the expression appearing in the stability condition (3.16). Therefore, $H(\overline{\zeta})$ is monotonically increasing (decreasing) for stable (unstable) solutions of the master equation. We can thus determine the stability of the solutions by a simple glance at the plot of the function $H(\overline{\zeta})$ (c.f. figure 1). In particular, the solutions of $H'(\overline{\zeta}) = 0$ mark the stability boundary of a given branch and correspond to the onset of the nonlocal tachyons.

The function $H(\overline{\zeta})$ also controls the monotonicity of the vacuum energy with respect to the effective coupling $\overline{\zeta}$. Starting from the general expression (2.17) we can write the vacuum energy density as

$$w = w_0 + f(w_0') - w_0'\, f'(w_0')\,, \qquad (3.19)$$

where all functions depend implicitly on $\overline{\zeta}$. Taking the derivative with respect to $\overline{\zeta}$ and using the definition of $H(\overline{\zeta})$ in (3.17) we find

$$w'(\overline{\zeta}) = w_0'(\overline{\zeta})\, H'(\overline{\zeta})\,. \tag{3.20}$$

If the condensate of the single-trace theory, $\langle \mathcal{O} \rangle \sim w_0'$, is a monotonic function of $\overline{\zeta}$ there is a correlation between the monotonicity of $H(\overline{\zeta})$ and that of the total vacuum energy.

It would be interesting to provide a more geometrical interpretation of these instabilities from the point of view of the NLST. The worldsheet interpretation of the multitrace insertions involves contact interactions between smooth worldsheets [12, 6]. The tachyonic modes described here should be associated to the condensation of these contact interactions, producing degenerating worldsheets. It would be interesting to elucidate the geometrical interpretation of these tachyons in spacetime. See [16] for recent results in this direction.

4. An explicit example

A simple model realizing these phenomena is the standard "approximation" of QCD in terms of Dp-branes (with $p < 5$) at finite temperature [14]. One starts by engineering a non-supersymmetric version of Yang-Mills theory in $d = p$ euclidean dimensions by considering the low-energy limit of a hot Dp-brane.

The important expansion parameter is the effective 't Hooft coupling of the YM$_d$ theory at the energy scale set by the temperature T of the SYM$_{d+1}$ theory on the hot D-brane (c.f. [17, 18]),

$$\lambda_d \sim g_s\, N \left(\sqrt{\alpha'}\right)^{d-3}\, T^{\,d-3}\,, \tag{4.1}$$

where g_s is the string coupling and α' the string's Regge slope. For $\lambda_d \ll 1$ perturbative Feynman diagrams give a good description, whereas for $\lambda_d \gg 1$ we can use the AdS/CFT dual in terms of the near-horizon metric of the black D-branes [17].

The single-trace action is given by

$$S_0 = \frac{N^2}{\lambda_d} \int d^d x\, \mathcal{L}_d\,, \qquad \mathcal{L}_d = \frac{T^{\,d-4}}{2N}\, \mathrm{tr}\, F^2 + \cdots\,, \tag{4.2}$$

where the dots stand for regularization artefacts at the temperature scale T or above (superpartners, higher-dimensional modes or string excitations). In what follows we shall simplify the notation by adopting units in which $T = 1$.

At $\lambda_d \gg 1$ the supergravity approximation yields an explicit value for the vacuum energy in terms of the free energy of the hot D-branes,

$$w_0 = -(5-d)\, C_d\, \lambda_d^{\frac{d-3}{5-d}}\,, \tag{4.3}$$

where C_d is a positive constant. From here one finds

$$\langle\, \mathcal{L}_d \rangle = w_0' = (d-3)\, C_d\, \lambda_d^{\frac{2}{5-d}}\,, \qquad G_2(0) = -w_0'' = \frac{2(d-3)}{5-d}\, C_d\, \lambda_d^{\frac{7-d}{5-d}}\,. \tag{4.4}$$

Hence, we have all the ingredients needed to consider multitrace deformations by a nonlinear function of the operator $\mathcal{L}_d(x)$,

$$S = S_0 + N^2 \int d^d x \, f(\, \mathcal{L}_d \,) \,. \tag{4.5}$$

Defining

$$\zeta = \frac{1}{\lambda_d} \,, \qquad H(\overline{\zeta}) = \overline{\zeta} - f' \left((d-3) C_d \, \overline{\zeta}^{\,\frac{2}{d-5}} \right) , \tag{4.6}$$

we have an effective single-trace model determined by the inverse 't Hooft coupling $\overline{\zeta}$, which acts as a curvature expansion parameter of the black Dp-brane metric. From this model we can calculate the condensate $\langle\, \mathcal{L}_d \rangle$ as a function of $\overline{\zeta}$, which in turn is determined by the master equation $\zeta = H(\overline{\zeta})$.

The multitrace deformation is trivial for $d = 3$ in agreement with the fact that the D3-brane free energy is independent of the dilaton in the leading supergravity approximation. Incidentally, we notice that $G_2(0) = -w_0'' < 0$ for $d < 3$. This violation of the positivity of the two-point function at zero momentum is presumably due to the non-Hermiticity of the effective lagrangian operator (4.2), which would be dominated at $\lambda_d \gg 1$ by the regularization artefacts. This fact renders the $d < 3$ models rather unphysical for the matters discussed here. Therefore, in the following we restrict attention to $d = 4$ and drop the d-dimensional subscript from all quantities. We also simplify the formulas by setting $C_4 = 1$ with an appropriate choice of coupling parameters.

The transition between the perturbative and supergravity descriptions occurs at the "correspondence line" of [19]. For the single-trace model it is given by $\zeta \sim 1$, which is perturbed by multitraces to $\overline{\zeta} \sim 1$ or, in terms of the original coupling

$$\zeta = H(1) = 1 - f'(1) \,. \tag{4.7}$$

We see that, depending on the sign of $f'(1)$, the multitraces increase or decrease the supergravity domain in ζ-space. We now discuss some specific choices for the multitrace perturbation.

4.1 Monomium perturbation

Choosing a multitrace deformation by a single positive power

$$f(z) = \xi \, \frac{z^{n+1}}{n+1} \,, \tag{4.8}$$

we have

$$H(\overline{\zeta}) = \overline{\zeta} - \frac{\xi}{\overline{\zeta}^{\,2n}} \tag{4.9}$$

and a vacuum energy density

$$w = -\overline{\zeta}^{\,-1} - \xi \, \frac{n}{n+1} \left(\overline{\zeta}^{\,-2} \right)^{n+1} . \tag{4.10}$$

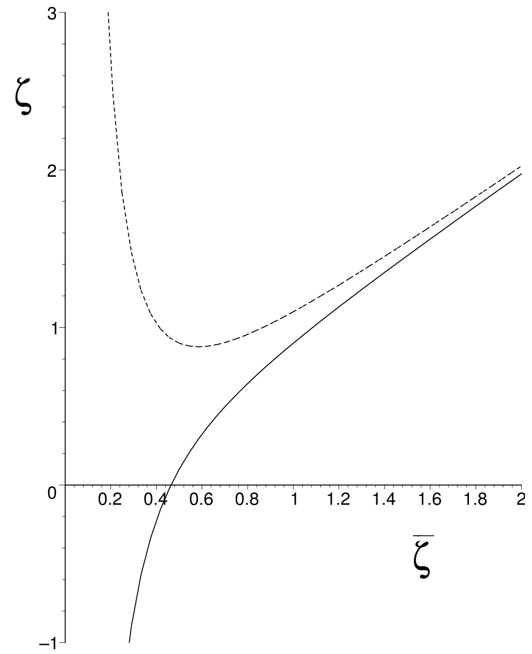

Figure 2: The function $H(\overline{\zeta})$ for the monomium perturbation with $n = 1$ and $\xi = 0.1$ (full line), $\xi = -0.1$ (dashed line).

For small ξ the behaviour is qualitatively equivalent to the single-trace theory near the correspondence line $\overline{\zeta} = 1$. However, there are gross differences deep in the supergravity regime $\overline{\zeta} \ll 1$.

For $\xi > 0$ we have well defined supergravity backgrounds with small curvature ($\overline{\zeta} \to 0$) and *small* and negative bare 't Hooft coupling ($\zeta \to -\infty$). Restricting ourselves to positive bare 't Hooft couplings we have a minimum value of $\overline{\zeta}$ given by $\overline{\zeta}_c = \xi^{\frac{1}{2n+1}}$. At this point the bare 't Hooft coupling diverges, i.e. $\zeta = H(\overline{\zeta}_c) = 0$.

On the other hand, for $\xi < 0$ the H-function has a minimum at $\overline{\zeta}_m = (2n|\xi|)^{\frac{1}{2n+1}}$ at which the bare 't Hooft coupling becomes maximal. This is a critical point of the master equation with the onset of a local instability for lower $\overline{\zeta}$. For $\zeta > \zeta_m$ there are two possible solutions of the master equation, but only the more curved one (larger $\overline{\zeta}$) is locally stable.

For either sign of ξ the dynamics differs significantly from the single-trace model when the inverse 't Hooft coupling reaches values of order $\zeta \sim |\xi|^{\frac{1}{2n+1}}$. Hence, for $|\xi| \sim 1$ the single-trace behaviour is pushed beyond the correspondence line. In particular, for $\xi \ll -1$ the supergravity regime loses all stable solutions.

4.2 Binomium perturbation

For a perturbation of the form

$$f(z) = \xi\, \frac{z^{n+1}}{n+1} + \eta\, \frac{z^{l+1}}{l+1}\,, \qquad n > l\,, \tag{4.11}$$

one has

$$H(\overline{\zeta}) = \overline{\zeta} - \frac{\xi}{\overline{\zeta}^{2n}} - \frac{\eta}{\overline{\zeta}^{2l}}\,. \tag{4.12}$$

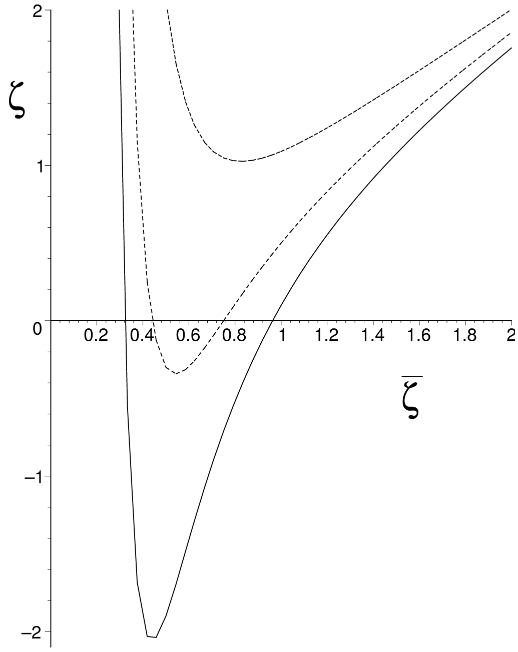

Figure 3: Binomium perturbation $n = 2$, $l = 1$, $\xi = -0.1$, $\eta = 1, 0.6, 0.01$ (the larger η corresponding to the full line).

There is no qualitative difference with the monomium if ξ and η have both the same sign. If the sign is different and $\xi < 0$, the shape is the same as in the monomium but the minimum is displaced to lower values of ζ, and can eventually cross the $\zeta = 0$ axis, removing the retrictions on the value of the 't Hooft parameter (c.f. figure 3). The amount of the displacement is related to the ratio $|\eta/\xi|$, growing with it up to a maximum value. If $\xi > 0$ a maximum and a minimum of $H(\overline{\zeta})$ appear if the ratio $|\eta/\xi|$ is large enough. Thus, a new unstable branch develops and there is a phase transition between the low-$\overline{\zeta}$ branch and the high-$\overline{\zeta}$ branch as ζ increases (c.f. figure 4).

4.3 More exotic perturbations

In the supergravity approximation one may also study non-polynomial "perturbations" at a formal level. For example, we can consider an analytic logarithmic perturbation $f(z) = \xi \log(\varepsilon z - 1)$, with $0 < \varepsilon < 1$, leading to

$$H(\overline{\zeta}) = \overline{\zeta} - \xi\, \varepsilon\, \frac{\overline{\zeta}^2}{\varepsilon - \overline{\zeta}^2}\,. \tag{4.13}$$

The function $f(\overline{\zeta})$ develops an imaginary part for $\overline{\zeta} > \sqrt{\varepsilon}$, so that we are led to the restriction $\overline{\zeta} < \sqrt{\varepsilon} < 1$. If $\varepsilon > 1$ this boundary goes beyond the supergravity regime. The $\xi < 0$ branch is locally stable, whereas the $\xi > 0$ branch is only stable up to a maximum value of $\overline{\zeta}$. This determines in turn a maximum value of ζ (or a minimum 't Hooft coupling).

An even more formal perturbation is given by the non-analytic function $f(z) = \xi/z$ with

$$H(\overline{\zeta}) = \overline{\zeta} + \xi\, \overline{\zeta}^2\,. \tag{4.14}$$

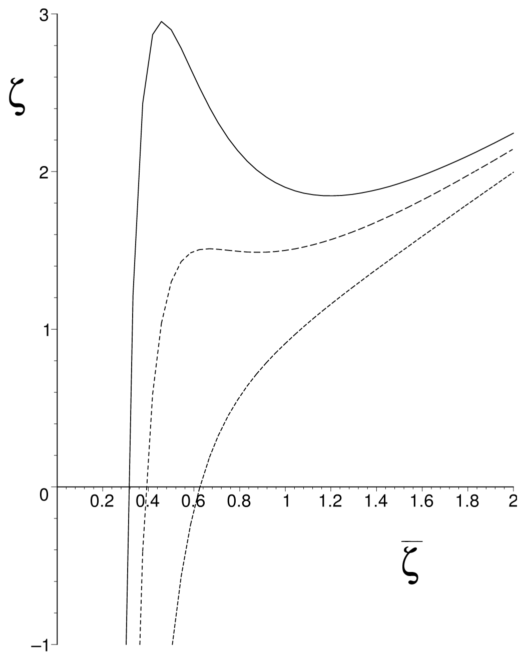

Figure 4: Binomium perturbation $n = 2$, $l = 1$, $\xi = 0.1$, $\eta = -1, -0.6, -0.01$. (the larger η corresponding to the full line).

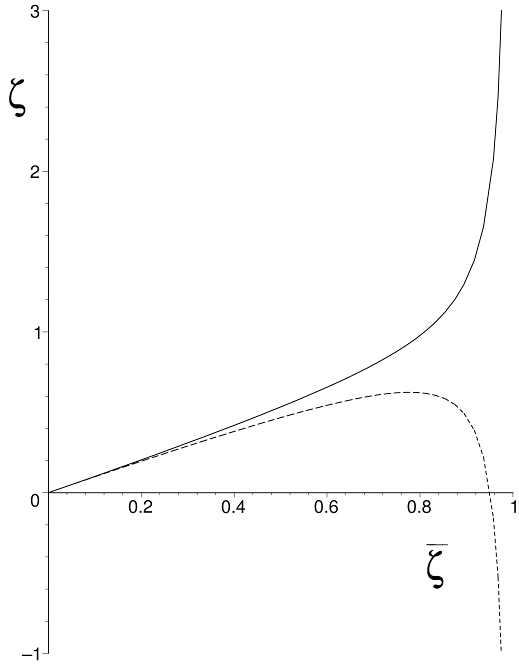

Figure 5: Logarithmic perturbation with $\varepsilon = 1$, $\xi = 0.1$ (dashed lines) and $\xi = -0.1$ (full lines). The branches at $\overline{\zeta} > 1$ are unphysical.

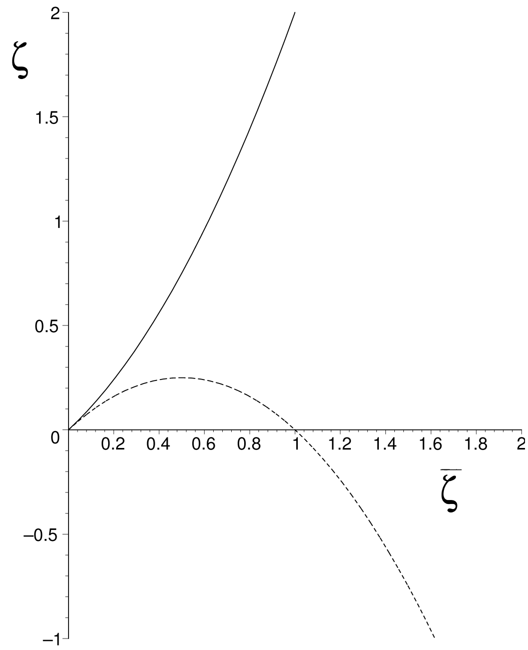

Figure 6: Simple pole perturbation,$f(z) = \xi/z$, with $\xi = 1$ (full line) and $\xi = -1$ (dashed line).

In this case, the perturbation by $1/\operatorname{tr} F^2$ does not make much sense in perturbation theory. However, the results are quite smooth in the supergravity approximation, corresponding to the limit of very large 't Hooft coupling. Indeed, from (4.14) we see that these models produce an *analytic* function $H(\overline{\zeta})$, so that they approach the single-trace theory for $\zeta \ll 1$. If $\xi < 0$ there is a maximum of the master equation, which means that the 't Hooft parameter should be higher than $4\,|\xi|$.

In general, we see that novel qualitative features triggered by multitrace couplings stay well within the supergravity approximation only as long as $|\xi| \ll 1$, with ξ a generic multitrace coupling. Since the master equation takes the form

$$\zeta = H(\overline{\zeta}) = \overline{\zeta} - f'\left(\overline{\zeta}^{\,-2}\right), \tag{4.15}$$

we find that deformations that have $f'(z)$ analytic around the origin produce important qualitative changes in the deep supergravity regime $\overline{\zeta} \to 0$. Conversely, singular deformations in perturbation theory, corresponding to singular $f'(z)$ at the origin, approach the single-trace theory in the extreme supergravity regime.

5. Multitraces and topology-changing phase transitions

Large-N phase transitions induced by multitrace couplings arise from the many possible solutions of the master equation $\zeta = H(\,\overline{\zeta}_i\,)$. However, in most cases considered so far the different solutions $\overline{\zeta}_i$ are continuously connected and the corresponding string backgrounds have the same topology when studied in the supergravity approximation. Large-N phase transitions with change of spacetime topology are known in the AdS/CFT framework, the most famous example being the Hawking–Page transition [13, 14], corresponding to a CFT on a finite-radius sphere. From the CFT point of view, the phase transition arises as a finite-size effect.

The interplay between multitrace-induced and topology-changing transitions is a interesting question that we address in this section. Fortunately, the example model based on hot D-branes does show topology-changing transitions when the Yang–Mills theory is compactified on a torus, so that we can carry on our study in a rather direct way. We start with a short review of the topology-changing transitions corresponding to finite-size effects of SYM models on toroidal compactifications.

5.1 Review of the single-trace case

Let us consider the compactification of the hot D4-brane on a $(4-p)$-dimensional torus of size L. In the perturbative description, the euclidean spacetime of the SYM model at finite temperature has the topology $\mathbb{R}^p \times \mathbf{S}^1_\beta \times \left(\mathbf{S}^1_L\right)^{4-p}$, with $\beta = 1/T$ and we take the supersymmetric spin structure on the torus of size L.

In the perturbative regime, $\lambda \ll 1$, the thermodynamics of the SYM_{4+1} theory changes character at $TL \sim 1$, from five-dimensional scaling of the entropy $S \sim T^4$ at $TL > 1$ to a p-dimensional scaling $S \sim T^{p-1}$ at $TL < 1$. At strong coupling $\lambda \gg 1$, the AdS/CFT correspondence incorporates this change of behaviour by a transition between topologically distinct backgrounds, both with the same asymptotic boundary conditions [18]. The first background is the near-horizon geometry of the original black D4-brane wrapped on the $(4-p)$-torus. This metric is T-dual to that of black Dp-branes, localized on the torus, but distributed uniformly over its volume, i.e. the so-called black brane "smeared" over $4-p$ transverse dimensions.

The vacuum energy per unit volume in $\mathbb{R}^p$ is the same for both T-dual metrics and is given by the thermodynamic free energy of the five-dimensional theory, i.e. we can write

$$w_0 = -\frac{1}{N^2\, V_p}\,\log\, \mathcal{Z}_0(T)\,, \tag{5.1}$$

where V_p is the volume in the noncompact $\mathbb{R}^p$ directions. In general, for a $(d+1)$-dimensional SYM theory at finite temperature T on a spatial volume V_d we have (c.f. [17, 18])

$$-\log\, \mathcal{Z}_0(T) = -N^2\, V_d\, C_d\, (g^2_{\rm eff} N)^{\frac{d-3}{5-d}}\, T^{\,\frac{9-d}{5-d}}\,, \tag{5.2}$$

where $g^2_{\rm eff}$ is the effective SYM coupling constant of mass dimension $3-d$. Considering now the particular case of D4-branes wrapped on the $\mathbf{T}^{4-p}$ torus, the effective coupling is $g^2_{\rm eff} N = \lambda/T$ and

$$w_{0,s} = -L^{4-p}\, C_4\, \lambda\, T^{\,4}\,. \tag{5.3}$$

With the same quantum numbers and asymptotic behaviour, one can consider the metric of Dp-branes fully localized on the $(4-p)$-torus. The corresponding vacuum energy is related to the thermodynamic free energy of the effective SYM theory in p euclidean

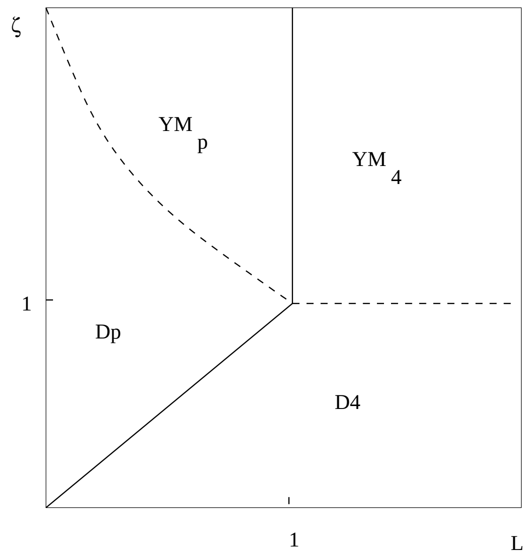

Figure 7: Phase diagram of the finite-temperature D4-brane at large N, as a function of the effective dimensionless coupling $\zeta = 1/\lambda$ and the size of the torus LT, in units of the temperature. Full lines denote finite-size localization transitions and dashed lines correspond to correspondence regions between supergravity and perturbative descriptions.

dimensions, with effective coupling $g_{\rm eff}^2 N = \lambda\, L^{p-4}/T$,

$$w_{0,\ell} = -N^2\, C_p \left(\frac{\lambda L^{p-4}}{T}\right)^{\frac{p-3}{5-p}}\; T^{\frac{9-p}{5-p}}\,. \tag{5.4}$$

The smeared metric dominates for large temperatures, whereas the localized metric takes over at low temperatures. The cross-over temperature is given by

$$1 = \frac{w_{0,s}}{w_{0,\ell}} = \frac{C_4}{C_p}\,(\lambda\, L\, T)^{\frac{2(p-4)}{p-5}}\;, \tag{5.5}$$

which defines the "localization curve"

$$\lambda \sim \frac{1}{L\,T}\,. \tag{5.6}$$

The transition between the smeared and the localized geometry is reminiscent of the Gregory–Laflamme instability [20]. They are, however, very different, since both backgrounds are locally stable in the near-horizon regime (they both have positive specfic heat [21]). Thus, we have a first-order phase transition between locally stable backgrounds.

We plot in figure 7 a phase diagram of the single-trace theory as a function of $\zeta = 1/\lambda$ and the dimensionless combination LT. The localization curve (5.6) continues at weak coupling as $LT = 1$. The correspondence lines separating the perturbative and the supergravity regimes are $\zeta = 1$ for large LT and $\zeta = (LT)^{p-4}$ for smaller values of LT.

5.2 The multitrace case

Since the multitrace deformation modifies the vacuum energy through (2.17) the phase transition curves change accordingly. Let us consider the simple case of a monomium perturbation of the form (4.8) in the $d = 4$ case. The "smeared" phase over the $(4-p)$-torus of size L has vacuum energy density

$$w_s = -L^{4-p}\,\overline{\zeta}_s^{\,-1} - L^{4-p}\,\xi\,\frac{n}{n+1}\left(\overline{\zeta}_s^{\,-2}\right)^{n+1}, \tag{5.7}$$

where we have chosen couplings so that $C_4 = 1$ and we use units with $T = 1$ throughout this section. $\overline{\zeta}_s$ denotes the selfconsistent coupling in the smeared phase, obeying the master equation

$$\zeta = \overline{\zeta}_s - \xi\,\overline{\zeta}_s^{\,-2n}\,. \tag{5.8}$$

On the other hand, the phase of localized black Dp-branes yields similar expressions after dimensional reduction to $\mathbb{R}^p$. The effective dimensionless coupling arising through the standard rule

$$\frac{1}{\lambda}\int d^4x \longrightarrow \frac{L^{4-p}}{\lambda}\int d^px \tag{5.9}$$

is given by

$$\zeta_{\rm eff} = L^{4-p}\,\zeta\,, \tag{5.10}$$

and similarly for $\overline{\zeta}$. The master equation of the localized phase is then

$$\zeta = \overline{\zeta}_\ell - \xi\left[(p-3)\,C_p\left(\overline{\zeta}_\ell\,L^{4-p}\right)^{\frac{2}{p-5}}\right]^n. \tag{5.11}$$

The resulting vacuum energy is given by

$$w_\ell = -(5-p)\,C_p\left(L^{4-p}\,\overline{\zeta}_\ell\right)^{\frac{3-p}{5-p}} - \xi\,\frac{n}{n+1}\left[(p-3)\,C_p\left(\overline{\zeta}_\ell\,L^{4-p}\right)^{\frac{2}{p-5}}\right]^{n+1}. \tag{5.12}$$

In general, we see that the multitrace perturbation is not qualitatively significant for

$$\zeta \gg |\xi|^{\frac{1}{2n+1}} \tag{5.13}$$

in the smeared phase. In the localized phase, the qualitative features are standard for

$$\zeta \gg L^{p-4}\,|\xi|^{\frac{5-p}{5-p+2n}}\,. \tag{5.14}$$

We shall further simplify the analysis by choosing $p = 3$, i.e. compactification on a single circle of size L. In this particular case the equations (5.11) and (5.12) governing the localized phase collapse to very simple expressions:

$$\zeta = \overline{\zeta}_\ell\,, \qquad w_\ell = -C_3\,. \tag{5.15}$$

The transition curve between the smeared and the localized phases in the supergravity regime follows from the equation

$$1 = \frac{w_s}{w_\ell} = \frac{L}{2C_3\,\overline{\zeta}_s}\left[1 + \frac{n}{n+1}\frac{\xi}{\overline{\zeta}_s^{\,2n+1}}\right]. \tag{5.16}$$

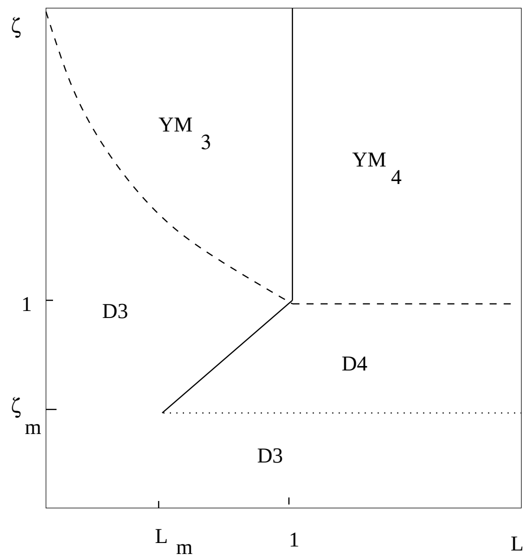

Figure 8: Phase diagram for the localization transition between D4-branes and D3-branes with a multitrace perturbation $\xi < 0$. The localization line terminates at (ζ_m, L_m) and the dotted line signals the local instability of the D4-branes, induced by the multitrace couplings.

For $\xi < 0$ we have a minimal value of $\overline{\zeta}_s$ for which the smeared solution is locally stable:

$$\overline{\zeta}_m = (2n\,|\xi|)^{\frac{1}{2n+1}}\,. \tag{5.17}$$

The main effect of this in the (ζ, L) phase diagram is the abrupt termination of the localization line (5.16) at the point (ζ_m, L_m), with

$$\zeta_m = \frac{2n+1}{2n}\,(2n\,|\xi|)^{\frac{1}{2n+1}}\,, \qquad L_m = \frac{2\,C_3\,(n+1)}{2n+1}\,(2n\,|\xi|)^{\frac{1}{2n+1}}\,. \tag{5.18}$$

Hence, part of the supergravity regime that was dominated by D4-branes is now covered by the D3-brane phase due to the local instabilities induced by the multitrace coupling (see figure 8).

For $\xi > 0$ the D4-brane phase is locally stable, but the condition $\zeta > 0$ still enforces a minimum value of $\overline{\zeta}_s$, given by $\overline{\zeta}_c^{\,2n+1} = \xi$. The critical length at this point is

$$L_c = \frac{n+1}{2n+1}\,2\,C_3\,\overline{\zeta}_c\,. \tag{5.19}$$

Expanding the master equation (5.8) and the localization equation (5.16) near the point $(\zeta = 0, L_c)$ we find

$$\zeta \approx \frac{2n+1}{n+1}\,\overline{\zeta}_c\left(1 - \frac{L_c}{L}\right) \tag{5.20}$$

in the vicinity of the $\zeta = 0$ axis. The resulting phase diagram is depicted in figure 9.

Thus, comparing with figure 7, we see that the multitraces tend to modify the structure of topology-changing phases in the extreme supergravity regime (low values of the effective coupling $\overline{\zeta}$).

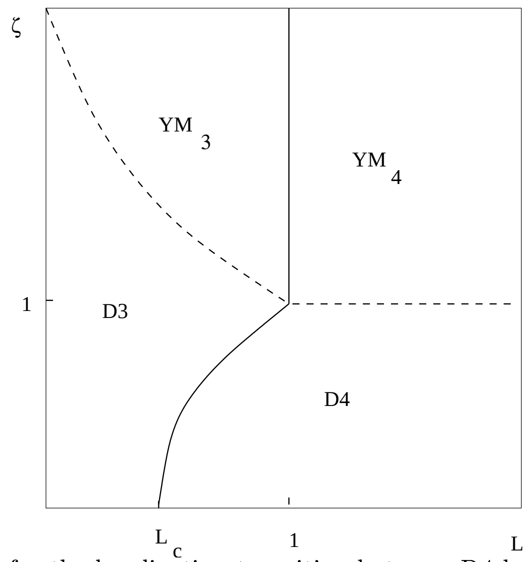

Figure 9: Phase diagram for the localization transition between D4-branes and D3-branes with a multitrace perturbation $\xi > 0$. The localization line intersects the $\zeta = 0$ axis at $L = L_c$, cutting off part of the D4-brane phase. In this case, there are no local instabilities.

6. Conclusions

We have studied multitrace perturbations of large-N master fields in theories with adjoint fields and non-vanishing mass gap. The introduction of suitable auxiliary fields allows us to discuss the systematics of the $1/N$ expansion at a formal level. At the same time, we were able to derive the "master equation" of the Hartree approximation (1.8), and define criteria for local stability of the master fields.

The example of hot D4-branes was studied in some detail. In particular, we find the conditions for the model to develop tachyons induced by multitrace operators. The qualitative features of many types of "master equations", including those of non-polynomial multitraces, were analyzed in the supergravity approximation, as well as the effects of these deformations on the thermodynamic phase diagrams at finite volume. In general, we find that the pattern of large-N phase transitions is strongly affected by the multitraces when the perturbation is of $O(1)$ in dimensionless units.

The main result of this paper is the relation between the global properties of the master equation, in particular the monotonicity of the function $H(\overline{\zeta})$ in (3.17), and the local stability of the master field. Since multitrace perturbations in the AdS/CFT context correspond to the nonlocal string theories of [6], we have found new classes of tachyonic instabilities of these models, entirely induced by the multitrace couplings. These instabilities arise despite the fact that, within the Hartree approximation, one can compute operator condensates in an auxiliary single-trace model (the background $\overline{X}$). It would be interesting to sharpen the connection between these instabilities and the so called "stringy exclusion principle" of [5].

Further lines of research suggested by our work include a more precise interpretation of the tachyons in the "bulk" spacetime description (as peculiar instabilities of $\overline{X}$) and the relation to supersymmetry breaking (in our examples, supersymmetry is broken to generate a mass gap, already at the level of the single-trace model). In particular, some type of UV/IR dictionary for the composite field $\sigma(x)$ would be required to understand how these instabilities proceed in the bulk. It should also be interesting to pursue the calculation of $1/N$ effects, starting with the one-loop determinants, for example along the lines of [22].

Acknowledgments

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A. Topological charge and multitrace perturbations

In the general discussion of section 2 we assumed the standard large-N scaling of one-point functions $\lim_{N\to\infty}\langle\mathcal{O}\rangle = O(1)$, with the normalization $\mathcal{O} \sim N^{-1}\,\mathrm{tr}\,F^n$. The purpose of this appendix is to point out an important exception of this rule, namely the case of the topological operator

$$\mathcal{Q} = \frac{1}{8\pi^2}\,\mathrm{tr}\,F\wedge F\,. \tag{A.1}$$

Notice the absence of the N^{-1} factor in front. Despite this enhanced normalization, the large-N one-point function $\langle\,\mathcal{Q}\rangle$ is still of $O(1)$, provided the θ angle is nonvanishing in the action

$$S_0 = \frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F^2 + \frac{i\theta}{8\pi^2}\int \mathrm{tr}\,F\wedge F\,. \tag{A.2}$$

Extracting an explicit power of N in the normalization of the topological action, we see that it is θ/N what plays the role of the 't Hooft coupling. Naively this is incompatible with 2π periodicity in θ, but it can be restored if the effective action develops multiple branches (c.f. [23])

$$\mathcal{Z}(\theta) = \exp\left[-N^2 W_0(\theta/N)\right] = \int DA\,e^{-S_0} = \sum_k \exp\left[-N^2\,V\,h\left(\frac{\theta+2\pi k}{N}\right)\right], \tag{A.3}$$

where V is the euclidean spacetime volume, and the function $h(y)$ is assumed to have a Taylor expansion with $O(1)$ coefficients in the large-N limit.

For each θ, the sum over k is dominated by the least-action branch in the infinite volume limit,

$$W_0(\theta/N) \approx \min{}_k\,V\,h\left(\frac{\theta+2\pi k}{N}\right). \tag{A.4}$$

The solution of the η' puzzle by Witten and Veneziano [24] requires that the function $h(y)$ behaves as $h(y) \sim y^2$ near the origin. More specifically, let us assume the result of ref. [25] in the AdS/CFT case and set

$$h(y) = \frac{1}{2}\, C\, y^2 + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{A.5}$$

It follows that the full theta-dependence of the vacuum energy is also of $O(1)$. For example, the one-point function of $\mathcal{Q}$ scales as

$$\langle\, \mathcal{Q}\rangle = \frac{i}{V}\partial_\theta \,\log\, \mathcal{Z}(\theta) = -iN^2 \frac{1}{N} h'\left(\frac{\theta + 2\pi k_c}{N}\right) \approx -i\, C\, (\theta + 2\pi k_c) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{A.6}$$

where k_c is the integer that minimizes (A.4). Therefore, the one-point function of the topological charge is of $O(1)$ in the large-N limit, despite its anomalous normalization as it appears in (A.1).

The connected two-point function is related to the topological susceptibility as

$$\overline{Q}^2_{\rm top} = \int_x \langle \mathcal{Q}(x)\mathcal{Q}(0)\rangle_c = \frac{1}{V}\int_x\int_y \langle \mathcal{Q}(x)\mathcal{Q}(y)\rangle_c = -\frac{1}{V}\,\partial^2_\theta\, \log\, \mathcal{Z}(\theta)\,, \tag{A.7}$$

and this in turn determines the constant C in (A.5),

$$\overline{Q}^2_{\rm top} = \frac{1}{V} N^2 \frac{1}{N^2} h''\left(\frac{\theta + 2\pi k}{N}\right) \approx C\,. \tag{A.8}$$

The peculiar scaling properties of the topological charge operator affect the mean field analysis of the multitrace perturbations. Consider the perturbed action

$$S = S_0 + \int_{\mathbb{R}^4} f\,(\mathcal{Q})\,. \tag{A.9}$$

Introducing the auxiliary fields and taking into account the assumed multibranched structure of the single-trace theory we find

$$\mathcal{Z}(\theta) = \sum_k \int D\,\sigma\, D\,\chi\, \exp\left[-N^2\Gamma\left(\frac{\theta + \chi + 2\pi k}{N}\right) - \int f(\sigma) + i\int \chi\,\sigma\right], \tag{A.10}$$

where the functional Γ/V equals the function h in (A.3) when evaluated on constant functions.

There is a saddle point for each of the branches, satisfying the equations

$$\chi_k = f'(\sigma_k)\,, \qquad \sigma_k = \langle\, \mathcal{Q}\rangle_{\overline{\theta}_k}\,, \tag{A.11}$$

where we have defined the effective theta angle $\overline{\theta}_k \equiv \theta + \chi_k + 2\pi k$, so that the master equation for each branch reads

$$\theta + 2\pi k = \overline{\theta}_k - f'\left(\langle\, \mathcal{Q}\rangle_{\overline{\theta}_k}\right). \tag{A.12}$$

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The quadratic fluctuation operator is

$$\mathcal{K}_{\mathcal{Q}} = \begin{pmatrix} \langle\, \mathcal{Q}\, \mathcal{Q} \rangle_{c,k} & -i \\ -i & f''(\langle\, \mathcal{Q} \rangle_k) \end{pmatrix}, \tag{A.13}$$

and its contribution to the vacuum energy in each branch is

$$\frac{1}{2}\,\mathrm{Tr}\log\left[1 + f''(\sigma_k)\langle \mathcal{Q}\mathcal{Q}\rangle_{c,k}\right]. \tag{A.14}$$

It contributes to the theta-dependence at the $O(1)$ order, just as the function $h(y)$ above. Hence, one must calculate the determinant in order to find the balance between the different branches.

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